

On summation of the Taylor series of the function $\frac{1}{1-z}$ by the theta summation method.

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Abstract

The family of the Taylor series $f_\varepsilon(z) = \sum_{0 \leq n < \infty} e^{-\varepsilon n^2} z^n$ is considered, where the parameter ε , which enumerates the family, runs over $]0, \infty[$. For each fixed $\varepsilon > 0$, this Taylor series converges locally uniformly with respect to $z \in \mathbb{C}$ and represents an entire function in z of zero order. The limiting behavior of the family $\{f_\varepsilon(z)\}_{0 < \varepsilon < \infty}$ is studied as $\varepsilon \rightarrow +0$. Let \mathcal{G} be the interior of the closed curve $\mathcal{C} = \{\zeta \in \mathbb{C} : \zeta = e^{|t|+it}, t \in [-\pi, \pi]\}$. It was shown that $\lim_{\varepsilon \rightarrow +0} f_\varepsilon(z) = 1/(1-z)$ for $z \in \mathcal{G}$ locally uniformly with respect to z . Moreover, $\overline{\lim}_{\varepsilon \rightarrow +0} |f_\varepsilon(z)| = \infty$ for $z \notin \mathcal{G}$.

Notation:

\mathbb{C} stands for the complex plane;

\mathbb{R} stands for the real axis;

\mathbb{Z} stands for the set of all integers;

\mathbb{T} stands for the unit circle: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$;

\mathbb{D} stands for the open unit disc: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;

$\overline{\mathbb{D}}$ stands for the closed unit disc: $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T} = \{z \in \mathbb{C} : |z| \leq 1\}$;

\mathbb{D}^- stands for the exterior of the unit circle \mathbb{T} : $\mathbb{D}^- = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$;

$\{z\}$ stands for the one-point set which consists of the point z .

1 Summation methods of Taylor Series

Let $f(z)$ be a function holomorphic in the unit disc \mathbb{D} . Such a function $f(z)$ can be expanded in the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (1.1)$$

The series in the right hand side of the equality (1.1) converges uniformly on every compact subset of the disk \mathbb{D} . Assume that the radius of convergence of this series is equal to one, that is

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1. \quad (1.2)$$

Then the function f can not be extended as a holomorphic function to any disc $\{z : |z| < R\}$ with $R > 1$. In other word, on the unit circle \mathbb{T} there is at least one singular point of the function f .

Assume moreover that the function f is holomorphic in some domain \mathcal{D} ,

$$\mathbb{D} \subset \mathcal{D} \subset \mathbb{C}.$$

Of course, the boundary $\partial\mathcal{D}$ of \mathcal{D} must intersect with \mathbb{T} : $\partial\mathcal{D} \cap \mathbb{T} \neq \emptyset$. Otherwise the radius of convergence of the Taylor series (1.1) will be greater than one.

The question arises. Can the Taylor series (1.1) be summed to the function f in some domain \mathcal{G} larger then the unit disc \mathbb{D} :

$$\mathbb{D} \subset \mathcal{G} \subseteq \mathcal{D}. \quad (1.3)$$

A summation method is determined by a sequence $\{\gamma_n(\varepsilon)\}_{0 \leq n < \infty}$, where for each $n = 0, 1, 2, 3, \dots$, $\gamma_n(\varepsilon)$ is a complex valued function defined for $0 < \varepsilon < \varepsilon_0$, $0 < \varepsilon_0 \leq +\infty$, and the following conditions are satisfied:

- a.
$$\sup_{\substack{0 < \varepsilon < \varepsilon_0 \\ 0 \leq n < \infty}} |\gamma_n(\varepsilon)| < \infty,$$
- b.
$$\lim_{\varepsilon \rightarrow +0} \gamma_n(\varepsilon) = 1 \quad \text{for each } n = 0, 1, 2, 3, \dots,$$
- c.
$$\lim_{n \rightarrow \infty} \sqrt[n]{|\gamma_n(\varepsilon)|} = 0 \quad \text{for each } \varepsilon \in (0, \varepsilon_0).$$

Such a sequence $\{\gamma_n(\varepsilon)\}_{0 \leq n < \infty}$ is said to be a *summing sequence*. Let $f_\varepsilon(z)$ be the function defined by the power series

$$f_\varepsilon(z) = \sum_{0 \leq n < \infty} \gamma_n(\varepsilon) a_n z^n. \quad (1.4)$$

The conditions (1.2) and c. ensure that the radius of convergence of the power series in (1.4) is equal to infinity. Thus the function $f_\varepsilon(z)$ is an entire function for each $\varepsilon > 0$. The equality (1.1) and the conditions (1.2), a. and b. ensure that

$$\lim_{\varepsilon \rightarrow +0} f_\varepsilon(z) = f(z) \quad (1.5)$$

locally uniformly for $z \in \mathbb{D}$.

If the limiting relation (1.5) holds locally uniformly in the domain \mathcal{G} , (1.3), than we say that the Taylor series of the function f is summable to the function f by the summation method $\{\gamma_n(\varepsilon)\}_{0 \leq n < \infty}$ in the domain \mathcal{G} .

The following three summing sequences are well known (see [1, Notes on Chapter VIII, §8.10]):

$$\gamma_n(\varepsilon) = \frac{\Gamma(1 + \varepsilon n)}{\Gamma(1 + n)}, \quad n = 0, 1, 2, \dots, \quad (1.6a)$$

$$\gamma_0(\varepsilon) = 1, \quad \gamma_n(\varepsilon) = e^{-\varepsilon n \log n}, \quad n = 1, 2, \dots, \quad (1.6b)$$

$$\gamma_n(\varepsilon) = \frac{1}{\Gamma(1 + \varepsilon n)}, \quad n = 0, 1, 2, \dots. \quad (1.6c)$$

Each of these three sequences sums the Taylor series $\sum_{0 \leq n < \infty} z^n$ of the function $f(z) = \frac{1}{1-z}$ to this function in the domain $\mathbb{C} \setminus [1, +\infty[$:

$$\frac{1}{1-z} = \lim_{n \rightarrow \infty} \gamma_n(\varepsilon) z^n, \quad \text{locally uniformly for } z \in \mathbb{C} \setminus [1, +\infty[. \quad (1.7)$$

The result (1.7) can be applied to the summing procedure (1.4)-(1.5) applied to an *arbitrary* function $f(z)$ whose Taylor series (1.1) has a positive radius of convergence. The domain \mathcal{G} , where the limiting relation (1.5) holds, is the so called *Mittag-Leffler star* of the function f . See [1, Chapter VIII, § 8.10, Theorem 135.]. (The Mittag-Leffler star of the function $\frac{1}{1-z}$ is the domain $\mathbb{C} \setminus [1, +\infty[$.)

2 Theta summation method. Convergence.

In this paper we discuss only one special summing sequence:

$$\gamma_n(\varepsilon) = e^{-\varepsilon n^2}, \quad n = 0, 1, 2, \dots \quad (2.1)$$

the summation method corresponding to the summing sequence (2.1) is said to be the theta summation method.

We apply the theta summation method the Taylor series $\sum_{0 \leq n < \infty} z^n$ of the function $f(z) = \frac{1}{1-z}$.

We succeeded in finding a precise answer to the following question: *for what* $z \in \mathbb{C} \setminus \{1\}$ *the limiting relation*

$$\frac{1}{1-z} = \lim_{\varepsilon \rightarrow +0} f_\varepsilon(z), \quad (2.2a)$$

holds, where

$$f_\varepsilon(z) \stackrel{\text{def}}{=} \sum_{0 \leq n < \infty} e^{-\varepsilon n^2} z^n. \quad (2.2b)$$

Lemma 2.1. *Given $\varepsilon > 0$, the series (2.2b) converges for every $z \in \mathbb{C}$ locally uniformly with respect to z . For each $\varepsilon > 0$, the function $f_\varepsilon(z)$ defined by this series is an entire function of zero order:*

$$\lim_{r \rightarrow \infty} \frac{\ln \ln M_{f_\varepsilon}(r)}{\ln r} = 0, \quad \text{where } M_{f_\varepsilon}(r) = \max_{z: |z| \leq r} |f_\varepsilon(z)|. \quad (2.3)$$

In contrast to the cases of the limiting relations (1.7) with $\gamma_n(\varepsilon)$ of the form (1.6), the set \mathcal{G} of those z , where the limiting relation (2.2) holds, is essentially smaller than the domain $\mathbb{C} \setminus [1, +\infty[$. (Nevertheless, still $\mathcal{G} \supset \mathbb{D}$, $\mathcal{G} \neq \mathbb{D}$.)

Starting point of our reasoning is the following Fourier representation of the summing sequence $\{e^{-\varepsilon n^2}\}_{0 \leq n < \infty}$:

$$e^{-\varepsilon n^2} = \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{+\infty} e^{-\xi^2/4\varepsilon} e^{in\xi} d\xi, \quad \varepsilon > 0. \quad (2.4)$$

Substituting (2.4) into (2.2b), we obtain the following representation for the function $f_\varepsilon(z)$, (2.2b),

$$f_\varepsilon(z) = \frac{1}{2\sqrt{\pi\varepsilon}} \sum_{0 \leq n < \infty} \int_{-\infty}^{+\infty} e^{-\xi^2/4\varepsilon} e^{in\xi} z^n d\xi, \quad \varepsilon > 0, \quad (2.5)$$

which holds for arbitrary $z \in \mathbb{C}$. For $z \in \mathbb{D}$, we can change order of summation and integration in (2.5). Thus

$$f_\varepsilon(z) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{+\infty} e^{-\xi^2/4\varepsilon} \frac{1}{1 - ze^{i\xi}} d\xi, \quad \varepsilon > 0, \quad z \in \mathbb{D}. \quad (2.6)$$

Splitting the integral in the right hand side of (2.6) and changing a variable, $\xi \rightarrow \pm\sqrt{\varepsilon}\xi$, we obtain

$$f_\varepsilon(z) = f_\varepsilon^+(z) + f_\varepsilon^-(z), \quad (2.7)$$

where

$$f_\varepsilon^+(z) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_0^{+\infty} e^{-\xi^2/4\varepsilon} \frac{1}{1 - ze^{i\xi}} d\xi, \quad \varepsilon > 0, \quad z \in \mathbb{D}, \quad (2.8a)$$

$$f_\varepsilon^-(z) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_0^{+\infty} e^{-\xi^2/4\varepsilon} \frac{1}{1 - ze^{-i\xi}} d\xi, \quad \varepsilon > 0, \quad z \in \mathbb{D}. \quad (2.8b)$$

Both the integrals in (2.8) are taken over the ray $[0, +\infty[$. From formulas (2.8) is evident that each of the functions $f_\varepsilon^+(z)$, $f_\varepsilon^-(z)$ is holomorphic in the unit disc \mathbb{D} . Moreover,

$$\lim_{\varepsilon \rightarrow +0} f_\varepsilon^+(z) = \frac{1}{2} \cdot \frac{1}{1 - z}, \quad (2.9a)$$

$$\lim_{\varepsilon \rightarrow +0} f_\varepsilon^-(z) = \frac{1}{2} \cdot \frac{1}{1 - z}, \quad (2.9b)$$

for $z \in \mathbb{D}$ locally uniformly in \mathbb{D} .

It turns out that each of the functions of the family $\{f_\varepsilon^+\}_{\varepsilon > 0}$ can be continued analytically from the unit disc \mathbb{D} to a domain \mathcal{G}^+ , $\mathcal{G}^+ \supset \mathbb{D}$, and each of the functions of the family $\{f_\varepsilon^-\}_{\varepsilon > 0}$ can be continued analytically from the unit

disc \mathbb{D} to a domain \mathcal{G}^- , $\mathcal{G}^- \supset \mathbb{D}$. We describe the domains \mathcal{G}^+ and \mathcal{G}^- as follows. Let \mathcal{S}^+ and \mathcal{S}^- be spiral-shaped curves:

$$\mathcal{S}^+ = \{\zeta \in \mathbb{C} : \zeta = e^{(1-i)t}, 0 \leq t < \infty\}, \quad (2.10a)$$

$$\mathcal{S}^- = \{\zeta \in \mathbb{C} : \zeta = e^{(1+i)t}, 0 \leq t < \infty\}. \quad (2.10b)$$

By definition,

$$\mathcal{G}^+ = \mathbb{C} \setminus \mathcal{S}^+, \quad \mathcal{G}^- = \mathbb{C} \setminus \mathcal{S}^-. \quad (2.11)$$

It is clear that the domains \mathcal{G}^+ and \mathcal{G}^- are simply connected, and

$$\mathbb{D} \subset \mathcal{G}^+, \quad \mathbb{D} \subset \mathcal{G}^-. \quad (2.12)$$

To see that the functions $f_\varepsilon^+(z), f_\varepsilon^-(z)$ can be continued analytically from the disc \mathbb{D} to the domains \mathcal{G}^+ and \mathcal{G}^- respectively, we modify the integral representations (2.8) of these functions rotating a ray of integration. For fixed $z \in \mathbb{D}$ and $\varepsilon > 0$, the function $e^{-\xi^2/4\varepsilon} \frac{1}{1 - ze^{i\xi}}$ which appears in (2.8a) is holomorphic with respect to ξ within the angular domain $0 \leq \arg \zeta \leq \frac{\pi}{4}$. Moreover for each fixed ϑ , $0 < \vartheta < \frac{\pi}{4}$, this function is fast decaying as $|\xi| \rightarrow \infty$ within the angular domain $0 \leq \arg \xi \leq \vartheta$:

$$\left| e^{-\xi^2/4\varepsilon} \frac{1}{1 - ze^{i\xi}} \right| \leq \frac{1}{1 - |z|} e^{-|\xi|^2 \cos 2\vartheta/4\varepsilon}, \quad z \in \mathbb{D}, \quad 0 \leq |\xi| < \infty, 0 \leq \arg \xi \leq \vartheta. \quad (2.13)$$

Therefore in (2.8a) we can rotate a ray of integration counterclockwise :

$$\begin{aligned} f_\varepsilon^+(z) &= \frac{e^{i\vartheta}}{2\sqrt{\pi\varepsilon}} \int_0^{+\infty} e^{-\xi^2 e^{2i\vartheta}/4\varepsilon} \frac{1}{1 - ze^{i\xi} e^{i\vartheta}} d\xi \\ &= \frac{e^{i\vartheta}}{2\sqrt{\pi}} \int_0^{+\infty} e^{-\xi^2 e^{2i\vartheta}/4} \frac{1}{1 - ze^{i\varepsilon\xi} e^{i\vartheta}} d\xi, \quad \varepsilon > 0, \quad z \in \mathbb{D}, \quad 0 \leq \vartheta < \frac{\pi}{4}. \end{aligned} \quad (2.14a)$$

Analogously in (2.8b) we can rotate a ray of integration clockwise :

$$\begin{aligned} f_\varepsilon^-(z) &= \frac{e^{-i\vartheta}}{2\sqrt{\pi}} \int_0^{+\infty} e^{-\xi^2 e^{-2i\vartheta}/4} \frac{1}{1 - ze^{i\varepsilon\xi} e^{-i\vartheta}} d\xi, \\ &\quad \varepsilon > 0, \quad z \in \mathbb{D}, \quad 0 \leq -\vartheta < \frac{\pi}{4}. \end{aligned} \quad (2.14b)$$

For $\vartheta : 0 < \vartheta < \frac{\pi}{2}$, let \mathcal{S}_ϑ^+ and \mathcal{S}_ϑ^- be spiral curves

$$\mathcal{S}_\vartheta^+ = \{\zeta \in \mathbb{C} : \zeta = e^{(\operatorname{tg} \vartheta - i)t}, 0 \leq t < \infty\}, \quad (2.15a)$$

$$\mathcal{S}_\vartheta^- = \{\zeta \in \mathbb{C} : \zeta = e^{(\operatorname{tg} \vartheta + i)t}, 0 \leq t < \infty\}, \quad (2.15b)$$

and let \mathcal{G}_ϑ^+ and \mathcal{G}_ϑ^- be the sets

$$\mathcal{G}_\vartheta^+ = \mathbb{C} \setminus \mathcal{S}_\vartheta^+, \quad \mathcal{G}_\vartheta^- = \mathbb{C} \setminus \mathcal{S}_\vartheta^-. \quad (2.16)$$

(In this notation, the sets \mathcal{S}^\pm introduced in (2.10)-(2.11) are $\mathcal{S}_{\pi/4}^\pm, \mathcal{G}_{\pi/4}^\pm$).

For $0 < \theta < \frac{\pi}{2}$, each of the sets $\mathcal{G}_\vartheta^\pm$ is a connected open set (a domain) in \mathbb{C} containing the unit disc:

$$\mathbb{D} \subset \mathcal{G}_\vartheta^+, \quad \mathbb{D} \subset \mathcal{G}_\vartheta^-, \quad 0 < \vartheta < \frac{\pi}{2}. \quad (2.17)$$

The curve \mathcal{S}_ϑ^+ is the boundary of the domain \mathcal{G}_ϑ^+ . When ξ runs over the positive half-axis $[0, +\infty[$, the point $e^{-i\varepsilon\xi}e^{i\vartheta}$ runs over the curve \mathcal{S}_ϑ^+ for every fixed $\varepsilon > 0$. Therefore for $z \in \mathcal{G}_\vartheta$ and $\xi \in [0, \infty[$, the value $|e^{-i\varepsilon\xi}e^{i\vartheta} - z|$ is bounded away from zero:

$$|e^{-i\varepsilon\xi}e^{i\vartheta} - z| \geq \text{dist}(z, \mathcal{S}_\vartheta^+) > 0, \quad z \in \mathcal{G}_\vartheta^+, \quad 0 \leq \xi < \infty.$$

(Here $\text{dist}(z, \mathcal{S}_\vartheta^+)$ is the distance from the point $z \in \mathcal{G}_\vartheta^+$ to the set \mathcal{S}_ϑ^+ .) Thus, for fixed $\vartheta \in [\vartheta, \frac{\pi}{2}[$ and $z \in \mathcal{G}_\vartheta^+$, the function $\frac{1}{1 - ze^{i\varepsilon\xi}e^{i\vartheta}} = 1 + \frac{z}{e^{-i\varepsilon\xi}e^{i\vartheta} - z}$ of the variable ξ is bounded on the positive half-axis $0 \leq \xi < \infty$:

$$\left| \frac{1}{1 - ze^{i\varepsilon\xi}e^{i\vartheta}} \right| \leq 1 + \frac{|z|}{\text{dist}(z, \mathcal{S}_\vartheta^+)}, \quad z \in \mathcal{G}_\vartheta^+, \quad 0 \leq \xi < \infty, \varepsilon > 0. \quad (2.18)$$

For $\vartheta : 0 < \vartheta < \frac{\pi}{4}$, and $\varepsilon > 0$, let us define

$$f_{\varepsilon, \vartheta}^+(z) = \frac{e^{i\vartheta}}{2\sqrt{\pi}} \int_0^{+\infty} e^{-\xi^2 e^{2i\vartheta}/4} \frac{1}{1 - ze^{i\varepsilon\xi}e^{i\vartheta}} d\xi, \quad z \in \mathcal{G}_\vartheta^+. \quad (2.19)$$

In view of the equality

$$|e^{-\xi^2 e^{2i\vartheta}/4}| = e^{-\xi^2 \cos 2\vartheta}, \quad 0 \leq \xi < \infty, \quad (2.20)$$

and the estimates (2.18), the function $e^{-\xi^2 e^{2i\vartheta}/4} \frac{1}{1 - ze^{i\varepsilon\xi}e^{i\vartheta}}$ which appears under the integral (2.19) admits the estimate

$$\left| e^{-\xi^2 e^{2i\vartheta}/4} \frac{1}{1 - ze^{i\varepsilon\xi}e^{i\vartheta}} \right| \leq \left(1 + \frac{|z|}{\text{dist}(z, \mathcal{S}_\vartheta^+)} \right) \cdot e^{-\xi^2 \cos 2\vartheta}, \quad 0 \leq \xi < \infty, \quad \varepsilon > 0. \quad (2.21)$$

Since the function $e^{-\xi^2 \cos 2\vartheta}$ is integrable:

$$\int_0^\infty e^{-\xi^2 \cos 2\vartheta} d\xi = \sqrt{\frac{\pi}{\cos 2\vartheta}}, \quad (2.22)$$

the integral in (2.19) exists. The function $f_{\varepsilon, \vartheta}^+(z)$ which is determined by means of this integral is well defined and holomorphic for $z \in \mathcal{G}_\vartheta^+$. In view of (2.14a),

$$f_\varepsilon^+(z) = f_{\varepsilon, \vartheta}^+(z) \quad \text{for } z \in \mathbb{D}. \quad (2.23)$$

Thus the function $f_{\varepsilon, \vartheta}^+$ is an analytic continuation of the function f_ε^+ from the unit disc \mathbb{D} to the domain \mathcal{G}_ϑ^+ .

From (2.19), (2.21), (2.22) we conclude that the family $\{f_{\varepsilon, \vartheta}^+\}_{\varepsilon > 0}$ is locally bounded in the domain \mathcal{G}_ϑ^+ , where the bound is uniform with respect to ε :

$$|f_{\varepsilon, \vartheta}^+(z)| \leq \sqrt{\frac{1}{\cos 2\vartheta}} \cdot \left(1 + \frac{|z|}{\text{dist}(z, \mathcal{S}_\vartheta^+)}\right), \quad z \in \mathcal{G}_\vartheta^+, \quad \varepsilon > 0. \quad (2.24)$$

In particular, the family $\{f_{\varepsilon, \vartheta}^+(z)\}_{\varepsilon > 0}$ is normal in the domain \mathcal{G}_ϑ^+ . According to (2.23), the limiting relation (2.9a) can be interpreted as

$$\lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \vartheta}^+(z) = \frac{1}{2(1-z)}, \quad z \in \mathbb{D}.$$

From here and from the normality of the family $\{f_{\varepsilon, \vartheta}^+(z)\}_{\varepsilon > 0}$ in \mathcal{G}_ϑ^+ it follows that

$$\lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \vartheta}^+(z) = \frac{1}{2(1-z)} \quad \text{for } z \in \mathcal{G}_\vartheta^+ \text{ locally uniformly.} \quad (2.25)$$

The relation (2.25) can be also obtained from (2.19) and the Lebesgue dominated convergence theorem.

Let us summarize the above-stated as

Lemma 2.2. *For every numbers ϑ and ε , $0 < \vartheta < \pi/4$, $0 < \varepsilon$, there exists a function $f_{\varepsilon, \vartheta}^+(\cdot)$ which possess the properties:*

1. *The function $f_{\varepsilon, \vartheta}^+(\cdot)$ is holomorphic in the domain \mathcal{G}_ϑ^+ and satisfies the estimate (2.24) there.*
2. *The function $f_{\varepsilon, \vartheta}^+(\cdot)$ is an analytic continuation of the function $f_\varepsilon^+(\cdot)$, (2.8a), from the unit disc \mathbb{D} to the domain \mathcal{G}_ϑ^+ , i.e. the equality (2.23) holds.*
3. *The limiting relation (2.25) holds.*

If $\vartheta = \pi/4$, the function $f_{\varepsilon, \pi/4}^+$ can not be defined by the integral (2.19) with $\theta = \pi/4$. This integral does not converges absolutely.¹ To define the function $f_{\varepsilon, \pi/4}^+$ in the domain $\mathcal{G}_{\pi/4}^+$, we glue together the functions $\{f_{\varepsilon, \vartheta}^+\}_{0 < \vartheta < \pi/4}$ into a single function.

¹We still can assign a meaning to the integral (2.19) (with $\theta = \pi/4$) by some regularization method. For example we can consider this integral as an improper integral. However even if we define the function $f_{\varepsilon, \pi/4}^+$ by an improper integral, we would be unable to prove the limiting relation (2.25) starting from such definition.

Remark 1. If $\vartheta', \vartheta'' \in]0, \pi/2[$, the functions $f_{\varepsilon, \vartheta'}^+$ and $f_{\varepsilon, \vartheta''}^+$ are defined and holomorphic in the domains $\mathcal{G}_{\vartheta'}^+$ and $\mathcal{G}_{\vartheta''}^+$, respectively. The unit disc \mathbb{D} is contained in the intersection of these domains: $\mathbb{D} \subset \mathcal{G}_{\vartheta'}^+ \cap \mathcal{G}_{\vartheta''}^+$. The functions $f_{\varepsilon, \vartheta'}^+$ and $f_{\varepsilon, \vartheta''}^+$ coincide on D : $f_{\varepsilon, \vartheta'}^+(z) = f_{\varepsilon, \vartheta''}^+(z) (= f_{\varepsilon}^+(z))$ for $z \in \mathbb{D}$. However if $\vartheta' \neq \vartheta''$, then the intersection $\mathcal{G}_{\vartheta'}^+ \cap \mathcal{G}_{\vartheta''}^+$ is not connected. Therefore we can only conclude that the functions $f_{\varepsilon, \vartheta'}^+$ and $f_{\varepsilon, \vartheta''}^+$ coincide on the connected component of the open set $\mathcal{G}_{\vartheta'}^+ \cap \mathcal{G}_{\vartheta''}^+$ that contains the origin. This circumstance complicates the reasoning a little. To carry out the reasoning smoothly, we introduce an auxiliary monotonic sequence of connected open sets O_n with compact closures $\overline{O_n}$ which sequence exhausts the domain $\mathcal{G}_{\pi/4}^+$.

A simple geometric construction² shows that the domain $\mathcal{G}^+ = \mathcal{G}_{\pi/4}^+$ can be represented as

$$\mathcal{G}_{\pi/4}^+ = \bigcup_{1 \leq n < \infty} O_n, \quad (2.26)$$

where the sequence $\{O_n\}_{1 \leq n < \infty}$ satisfies the conditions:

1. Each O_n is an open set;
2. The closure $\overline{O_n}$ of the set O_n is a compact set which is contained in the domain $\mathcal{G}_{\pi/4}^+$:

$$\overline{O_n} \text{ is a compact set, } \overline{O_n} \subset \mathcal{G}_{\pi/4}^+; \quad (2.27)$$

3. Each O_n is a connected set.
4. The sequence $\{O_n\}_{1 \leq n < \infty}$ increases:

$$O_1 \subseteq O_2 \subseteq O_3 \dots; \quad (2.28)$$

5. Every set O_n , $n = 1, 2, 3, \dots$, contains the disc $\mathbb{D}_{1/2}$, $\mathbb{D}_{1/2} = \{\zeta \in \mathbb{C} : |\zeta| < 1/2\}$:

$$\mathbb{D}_{1/2} \subset O_n, \quad n = 1, 2, 3, \dots \quad (2.29)$$

Let us choose and fix such a sequence of sets $\{O_n\}_{1 \leq n < \infty}$.

For $\theta \in]0, \pi/2[$, the domain \mathcal{G}_{θ}^+ depends on θ continuously, where the convergence of domains is the kernel convergence in the sense of Caratheodory. (Regarding the notion of kernel convergence, we refer to [2, section 1.4].) The relation $\lim_{\theta \rightarrow \pi/4-0} \mathcal{G}_{\theta} = \mathcal{G}_{\pi/4}$ means, in particular, that for every compact set K , $K \in \mathcal{G}_{\pi/4}$, there exists ϑ_K , $0 < \vartheta_K < \pi/4$, such that $K \subset \mathcal{G}_{\theta}$ for $\vartheta : \vartheta_K \leq \vartheta < \pi/4$.

²Such a construction can be done by many different ways. We omit a formal geometric construction of the sequence $\{O_n\}_{1 \leq n < \infty}$.

By choosing the set $\overline{O_n}$ as K , (2.27), we conclude that there exists ϑ_n , $0 < \theta_n < \pi/4$, such that

$$\overline{O_n} \subset \mathcal{G}_{\vartheta_n}^+. \quad (2.30)$$

As it was stated in Lemma 2.2, the function $f_{\varepsilon, \theta_n}^+$ is well defined and holomorphic in the domain $\mathcal{G}_{\vartheta_n}^+$. In particular, the function $f_{\varepsilon, \theta_n}^+$ is well defined and holomorphic in the domain O_n .

If $n_1 < n_2$, then the functions $f_{\varepsilon, \theta_{n_1}}^+$ and $f_{\varepsilon, \theta_{n_2}}^+$ are defined and holomorphic in the domains $\mathcal{G}_{\vartheta_{n_1}}^+$ and $\mathcal{G}_{\vartheta_{n_2}}^+$ respectively. Since

$$f_{\varepsilon, \theta_{n_1}}^+(z) = f_{\varepsilon, \theta_{n_2}}^+(z) \quad (= f_{\varepsilon}^+(z)) \quad \text{for } z \in \mathbb{D}_{1/2}$$

and $\mathbb{D}_{1/2} \subset O_{n_1} \subset O_{n_2}$, we conclude that

$$f_{\varepsilon, \theta_{n_1}}^+(z) = f_{\varepsilon, \theta_{n_2}}^+(z) \quad \text{for } z \in O_{n_1}, \quad n_1 < n_2. \quad (2.31)$$

In view of (2.26), (2.28) and (2.31), the sequence of functions $\{f_{\varepsilon, \theta_n}^+\}_{1 \leq n < \infty}$ can be glued together into a single function, which is defined on the set $\bigcup_n O_n = \mathcal{G}_{\pi/4}^+$.

We denote this function by $f_{\varepsilon, \pi/4}^+$:

$$f_{\varepsilon, \pi/4}^+(z) = f_{\varepsilon, \vartheta_n}^+(z) \quad \text{for } z \in O_n, \quad n = 1, 2, 3, \dots \quad (2.32)$$

The equalities (2.26), (2.32) serve as a *definition* of the function $f_{\varepsilon, \pi/4}^+$ in the domain $\mathcal{G}_{\pi/4}^+$. By virtue of (2.31), this definition is non-contradictory.

Lemma 2.3. *For every $\varepsilon, \varepsilon > 0$, there exists a function $f_{\varepsilon, \pi/4}^+$ such that*

1. *The function $f_{\varepsilon, \pi/4}^+(\cdot)$ is holomorphic in the domain $\mathcal{G}_{\varepsilon/4}^+$.*
2. *The function $f_{\varepsilon, \pi/4}^+(\cdot)$ is an analytic continuation of the function $f_{\varepsilon}^+(\cdot)$, which was defined by (2.8a), from the unit disc \mathbb{D} to the domain $\mathcal{G}_{\pi/4}^+$, i.e. the equality*

$$f_{\varepsilon, \pi/4}^+(z) = f_{\varepsilon}^+(z) \quad (2.33)$$

holds for every $z \in \mathbb{D}$.

3. *The functional family $\{f_{\varepsilon, \pi/4}^+\}_{0 < \varepsilon < \infty}$ is locally bounded in $\mathcal{G}_{\pi/4}^+$. In other words, for each compact set K , $K \subset \mathcal{G}_{\pi/4}^+$, the estimate*

$$|f_{\varepsilon, \pi/4}^+(z)| \leq C_K^+, \quad \forall z \in K, \quad (2.34)$$

holds, where the value $C_K^+ < \infty$ does not depend on ε .

4. The limiting relation

$$\lim_{\varepsilon \rightarrow +0} f_{\varepsilon, \pi/4}^+(z) = \frac{1}{2(1-z)}, \quad \forall z \in \mathcal{G}_{\pi/4}^+, \quad (2.35)$$

holds. In (2.35), the limit is locally uniform with respect to $z \in \mathcal{G}_{\pi/4}^+$.

Proof. 1. Taking into account (2.32), (2.30) and the holomorphy of the function $f_{\varepsilon, \vartheta_n}^+$ on the domain $\mathcal{G}_{\vartheta_n}^+$, we conclude that the function $f_{\varepsilon, \pi/4}^+$ is holomorphic on the domain O_n . In view of (2.26), the function $f_{\varepsilon, \pi/4}^+$ is holomorphic on the domain $\mathcal{G}_{\pi/4}^+$.

2. By virtue of (2.32), (2.29) and (2.23), the equality (2.33) holds for every $z \in \mathbb{D}_{1/2}$. Since both functions $f_{\varepsilon, \pi/4}^+$ and f_{ε}^+ are holomorphic on \mathbb{D} , the equality (2.33) holds for every $z \in \mathbb{D}$.

3. Let K be a compact set, $K \subset \mathcal{G}_{\pi/4}^+$. In view of (2.26), $K \subset \bigcup_{1 \leq n < \infty} O_n$. In view of (2.28), $K \subset O_{n_0}$ for some n_0 . All the more, $K \subset \mathcal{G}_{\vartheta_{n_0}}^+$, (2.30). In particular,

$$\text{dist}(K, \mathcal{S}_{\vartheta_{n_0}}^+) > 0, \quad (2.36)$$

where $\text{dist}(K, \mathcal{S}_{\vartheta_{n_0}}^+)$ is the distance from the set K to the boundary $\mathcal{S}_{\vartheta_{n_0}}^+$ of the domain $\mathcal{G}_{\vartheta_{n_0}}^+$. The equality (2.32) (with $n = n_0$) and the estimate (2.24) imply the estimate (2.34) with

$$C_K = \sqrt{\frac{1}{\cos 2\vartheta_{n_0}}} \cdot \left(1 + \frac{\max_{\zeta \in K} |\zeta|}{\text{dist}(K, \mathcal{S}_{\vartheta_{n_0}}^+)} \right).$$

4. Let K be a compact set, $K \subset \mathcal{G}_{\pi/4}^+$. As we saw, there exists n_0 such that $K \subset O_{n_0} \subset \mathcal{G}_{\vartheta_{n_0}}^+$. According to Lemma 2.2, $\lim_{\varepsilon \rightarrow +0} f_{\varepsilon, \vartheta_{n_0}}^+(z) = \frac{1}{2(1-z)}$ for each $z \in K$. Moreover, this limiting relation holds uniformly with respect to $z \in K$. In view of (2.32), $f_{\varepsilon, \pi/4}^+(z) = f_{\varepsilon, \vartheta_{n_0}}^+(z)$ for $z \in K$. \square

The same reasoning can be carried out for the function f_{ε}^- .

Lemma 2.4. For every $\varepsilon, \varepsilon > 0$, there exists a function $f_{\varepsilon, \pi/4}^-$ such that

1. The function $f_{\varepsilon, \pi/4}^-(.)$ is holomorphic in the domain $\mathcal{G}_{\pi/4}^-$.
2. The function $f_{\varepsilon, \pi/4}^-(.)$ is an analytic continuation of the function $f_{\varepsilon}^-(.)$, which was defined by (2.8b), from the unit disc \mathbb{D} to the domain $\mathcal{G}_{\pi/4}^-$, i.e. the equality

$$f_{\varepsilon, \pi/4}^-(z) = f_{\varepsilon}^-(z) \quad (2.37)$$

holds for every $z \in \mathbb{D}$.

3. The functional family $\{f_{\varepsilon, \pi/4}^-\}_{0 < \varepsilon < \infty}$ is locally bounded in $\mathcal{G}_{\pi/4}^-$. In other words, for each compact set K , $K \subset \mathcal{G}_{\pi/4}^-$, the estimate

$$|f_{\varepsilon, \pi/4}^-(z)| \leq C_K^-, \quad \forall z \in K, \quad (2.38)$$

holds, where the value $C_K^- < \infty$ does not depend on ε .

4. The limiting relation

$$\lim_{\varepsilon \rightarrow +0} f_{\varepsilon, \pi/4}^-(z) = \frac{1}{2(1-z)}, \quad \forall z \in \mathcal{G}_{\pi/4}^-, \quad (2.39)$$

holds. In (2.39), the limit is locally uniform with respect to $z \in \mathcal{G}_{\pi/4}^-$.

We denote by \mathcal{G} the connected component of the open set $\mathcal{G}_{\pi/4}^+ \cap \mathcal{G}_{\pi/4}^-$, which contains the origin. The set \mathcal{G} admits an explicit description.

Definition 1. Let \mathcal{C} be a closed curve which parametric representation is

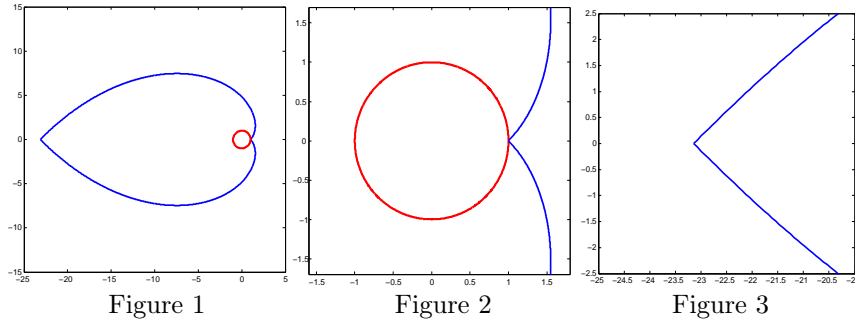
$$\mathcal{C} = \{\zeta \in \mathbb{C} : \zeta = e^{|t|+it}, \text{ where } t \text{ runs over } [-\pi, +\pi]\}. \quad (2.40)$$

The domain \mathcal{G} is the interior of the curve \mathcal{C} .

Remark 2. It is worthy to mention that the domain \mathcal{G} contains the open unit disc, more precisely

$$\mathbb{D} \setminus \{1\} \subset \mathcal{G}. \quad (2.41)$$

The heart-shaped curve \mathcal{C} is plotted by a solid blue line in the Figure 1 below. The unit circle \mathbb{T} is plotted by the solid red line. The curve \mathcal{C} intersects the real axis at the points with coordinates $(1, 0)$ and $(-e^\pi, 0)$. ($e^\pi = 23.140692632779267 \dots$) The fragments of this curve near these points are plotted in the Figures 2 and 3.



In the domain \mathcal{G} , both functions $f_{\varepsilon, \pi/4}^+$ and $f_{\varepsilon, \pi/4}^-$ are defined and holomorphic. Hence, the sum $f_{\varepsilon, \pi/4}^+(z) + f_{\varepsilon, \pi/4}^-(z)$ of these functions is defined and holomorphic on \mathcal{G} . According to (2.33), (2.37), and (2.7),

$$f_{\varepsilon, \pi/4}^+(z) + f_{\varepsilon, \pi/4}^-(z) = f_\varepsilon(z), \quad \forall z \in \mathbb{D},$$

where f_ε is defined by (2.2b). The function $f_{\varepsilon,\pi/4}^+ + f_{\varepsilon,\pi/4}^-$ is holomorphic in \mathcal{G} , the function $f_\varepsilon(z)$ is an entire function. Therefore

$$f_{\varepsilon,\pi/4}^+(z) + f_{\varepsilon,\pi/4}^-(z) = f_\varepsilon(z), \quad \forall z \in \mathcal{G}. \quad (2.42)$$

The following statement follows from the last equality and from the properties³ of the functional families $\{f_{\varepsilon,\pi/4}^\pm\}_{0 < \varepsilon < \infty}$.

Theorem 1. *For $\varepsilon > 0$, let $f_\varepsilon(z)$ be the function which is defined as the sum of the power series (2.2b). Let \mathcal{G} be the domain introduced in Definition 1. Then*

1. *For each $\varepsilon > 0$, the function f_ε is an entire function of order zero.*
2. *The functional family $\{f_\varepsilon\}$ is locally bounded in \mathcal{G} . This means that for every compact subset K of the domain \mathcal{G} , $K \subseteq G$, the inequality*

$$|f_\varepsilon(z)| \leq C_K, \quad \forall z \in K, \quad \forall \varepsilon > 0, \quad (2.43)$$

holds, where $C_K < \infty$ is a constant which does not depend on z and ε .

3. *The limiting relation holds*

$$\lim_{\varepsilon \rightarrow +0} f_\varepsilon(z) = \frac{1}{1-z}, \quad \text{for each } z \in \mathcal{G}. \quad (2.44)$$

The convergence $f_\varepsilon(z)$ to $\frac{1}{1-z}$ is locally uniform with respect to $z \in \mathcal{G}$.

3 Theta summation method. Divergence.

Theorem 2. *Let the function f_ε and the domain \mathcal{G} be the same that in Theorem 1.*

Then for every $z \notin \mathcal{G}$ the limiting relation

$$\overline{\lim}_{\varepsilon \rightarrow +0} |f_\varepsilon(z)| = \infty \quad (3.1)$$

holds.

It is clear that

$$\lim_{\varepsilon \rightarrow +0} \sum_{-\infty < n < 0} e^{-\varepsilon n^2} z^n = \sum_{-\infty < n < 0} z^n = \frac{z}{z-1}, \quad \forall z \in \mathbb{D}^-. \quad (3.2)$$

For $\varepsilon > 0$ and $z \in \mathbb{C} \setminus 0$, let

$$h_\varepsilon(z) = \sum_{-\infty < n < \infty} e^{-\varepsilon n^2} z^n \quad (3.3)$$

³ These properties were summarized in Lemmas 2.3, 2.4.

For each $\varepsilon > 0$, the function $h_\varepsilon(\cdot)$ is holomorphic in the domain $\mathbb{C} \setminus 0$ and satisfies the relation

$$h_\varepsilon(z) = h_\varepsilon(z^{-1}). \quad (3.4)$$

Definition 2. *Let*

$$\mathcal{V}_f = \{z \in \mathbb{C} : \overline{\lim}_{\varepsilon \rightarrow +0} |f_\varepsilon(z)| = \infty\}, \quad (3.5a)$$

$$\mathcal{V}_h = \{z \in \mathbb{C} : \overline{\lim}_{\varepsilon \rightarrow +0} |h_\varepsilon(z)| = \infty\}, \quad (3.5b)$$

The sets \mathcal{V}_f and \mathcal{V}_h are said to be the divergence set for the functional family $\{f_\varepsilon\}_{\varepsilon>0}$ and the divergence set for the functional family $\{h_\varepsilon\}_{\varepsilon>0}$ respectively.

Remark 3. *Theorem 2 is equivalent to the relation*

$$\mathbb{C} \setminus \mathcal{G} = \mathcal{V}_f. \quad (3.6)$$

Remark 4. *In view of (3.4), the set \mathcal{V}_h is invariant with respect to the transformation $z \rightarrow z^{-1}$: $z \in \mathcal{V}_h$ if and only $z^{-1} \in \mathcal{V}_h$.*

In view of (2.44) and (2.41),

$$\mathcal{V}_f \subset \{1\} \cup \mathbb{D}^-. \quad (3.7)$$

It is clear that

$$\{1\} \subset \mathcal{V}_f \cap \mathcal{V}_h. \quad (3.8)$$

Because of (3.7), (3.8), the equality

$$h_\varepsilon(z) = f_\varepsilon(z) + \sum_{-\infty < n < 0} e^{-\varepsilon n^2} z^n, \quad \forall z \in \mathbb{C}, \quad \forall \varepsilon > 0,$$

and (3.2), the following statement is evident.

Lemma 3.1. *For the divergency sets \mathcal{V}_f and \mathcal{V}_h , the following relation holds:*

$$\mathcal{V}_f = \{1\} \cup (\mathcal{V}_h \cap \mathbb{D}^-). \quad (3.9)$$

To find the divergence set \mathcal{V}_h , we use the equality

$$h_\varepsilon(z) = H_\varepsilon\left(\frac{\ln z}{2i}\right), \quad \forall z \in \mathbb{C} \setminus 0, \quad (3.10)$$

where

$$H_\varepsilon(\zeta) = \sqrt{\frac{\pi}{\varepsilon}} \sum_{-\infty < n < \infty} e^{-(\zeta - n\pi)^2/\varepsilon}, \quad \varepsilon > 0, \quad \zeta \in \mathbb{C}. \quad (3.11)$$

(The converges of the series in the right hand side of (3.11) is explained in Corollary 1 below.) The equality (3.10), where the functions $h_\varepsilon(\cdot)$ and $H_\varepsilon(\cdot)$ are define by (3.3) and (3.11), is a particular case of the Jacobi transformation for the theta function $\vartheta_3(z|\tau)$. (See [3, Chapter XXI, sect. 21.51], the formula before Example 1 there.) However the equality (3.10) can be obtained without any reference to the theory of theta functions, but by using the Poisson summation formula. (Concerning the Poisson summation formula we refer to [4, Chapter 2, sect. 7.5], pages 111-112.)

The function $H_\varepsilon(\zeta)$ is a periodic function with respect to ζ with a period π and also is an even function:

$$H_\varepsilon(\zeta + \pi) \equiv H_\varepsilon(\zeta), \quad H_\varepsilon(\zeta) \equiv H_\varepsilon(-\zeta). \quad (3.12)$$

Therefore the function $H_\varepsilon\left(\frac{\ln z}{2i}\right)$ is a single-valued function of z . In particular, this function does not depend on a choice of branch of $\ln z$.

Definition 3. *Let*

$$\mathcal{V}_H = \{\zeta \in \mathbb{C} : \overline{\lim}_{\varepsilon \rightarrow 0} |H_\varepsilon(\zeta)| = \infty\}. \quad (3.13)$$

The set \mathcal{V}_H is said to be the divergence set for the functional family $\{H_\varepsilon\}_{\varepsilon > 0}$.

In view of (3.12), the set \mathcal{V}_H possess the properties

$$\mathcal{V}_H + k\pi = \mathcal{V}_H, \quad \forall k \in \mathbb{Z}, \quad -\mathcal{V}_H = \mathcal{V}_H. \quad (3.14)$$

This means that if $\zeta \in \mathcal{H}$, then $(\zeta + k\pi) \in \mathcal{H}$ for any $k \in \mathbb{Z}$ and $(-\zeta) \in \mathcal{H}$.

The equality (3.10) for the functions $h_\varepsilon(\cdot)$ and $H_\varepsilon(\cdot)$ implies the following statement:

Lemma 3.2. *For the divergency sets \mathcal{V}_f and \mathcal{V}_h , the following relation⁴ holds:*

$$\mathcal{V}_h = \exp\{2i\mathcal{V}_H\} \quad (3.15)$$

Let us study the series in the right hand side of (3.11).

Lemma 3.3. *Let K be a compact subset of the complex plane \mathbb{C} . There exists a number $N = N(K)$, $1 \leq N(K) < \infty$, which depends only on K but not on ζ such that the inequality*

$$\operatorname{Re}(\zeta - n\pi)^2 \geq (n\pi)^2/2 \quad (3.16)$$

holds for every $\zeta \in K$ and for every $n \in \mathbb{Z}$ such that $|n| > N(K)$.

⁴ The relation (3.15) means that $\zeta \in \mathcal{V}_H$ if and only if $z = e^{2i\zeta} \in \mathcal{V}_h$.

Proof. Since $(\zeta - n\pi)^2 = (n\pi)^2(1 - \zeta/(n\pi))^2$, the equality holds

$$\operatorname{Re}(\zeta - n\pi)^2 = (n\pi)^2 \cdot \operatorname{Re}(1 - \zeta/(n\pi))^2.$$

If $a \in \mathbb{C}$, $|a| < 1/4$, then $\operatorname{Re}(1 - a)^2 \geq 1/2$. Indeed, $(1 - a)^2 = 1 + |a|^2 - 2\operatorname{Re} a$. Hence $\operatorname{Re}(1 - a)^2 \geq (1 - 2|a|) \geq 1/2$ if $|a| < 1/4$. Thus the assertion of Lemma holds with

$$N(K) = 4 \max_{\zeta \in K} |\zeta|/\pi. \quad (3.17)$$

□

Corollary 1. *The series in the right hand side of (3.11) converges for every $\zeta \in \mathbb{C}$. The convergence of this series is locally uniform with respect to ζ .*

Given $\zeta \in \mathbb{C}$, we split the series in the right hand side of (3.11) in two series:

$$H_\varepsilon(\zeta) = H_\varepsilon^1(\zeta) + H_\varepsilon^2(\zeta), \quad (3.18)$$

where

$$H_\varepsilon^j(\zeta) = \sqrt{\frac{\pi}{\varepsilon}} \sum_{n \in Z_j(\zeta)} e^{-(\zeta - n\pi)^2/\varepsilon}, \quad j = 1, 2, \quad \varepsilon > 0. \quad (3.19)$$

and

$$Z_1(\zeta) = \{n \in \mathbb{Z} : \operatorname{Re}(\zeta - n\pi)^2 \leq 0\}, \quad Z_2(\zeta) = \{n \in \mathbb{Z} : \operatorname{Re}(\zeta - n\pi)^2 > 0\}. \quad (3.20)$$

It is clear that $Z_1(\zeta) \cap Z_2(\zeta) = \emptyset$, $Z_1(\zeta) \cup Z_2(\zeta) = \mathbb{Z}$. From Lemma 3.3 it follows that $n \in Z_2(\zeta)$ if $|n| > 4|\zeta|/\pi$. So the set $Z_1(\zeta)$ is always finite, may be empty.

Remark 5. *If the set $Z_1(\zeta)$ is empty, we set $H_\varepsilon^1(\zeta) \equiv 0$. So the equality (3.18) holds always, whatever the set $Z_1(\zeta)$ is, empty or not.*

Lemma 3.4. *Whatever $\zeta \in \mathbb{C}$ is, the equality*

$$\lim_{\varepsilon \rightarrow +0} H_\varepsilon^2(\zeta) = 0. \quad (3.21)$$

holds.

Proof. Since $\operatorname{Re}(\zeta - n\pi)^2 > 0$ for every $n \in Z_2(\zeta)$, there exists the constant $c(\zeta) > 0$ such that $\operatorname{Re}(\zeta - n\pi)^2 \geq c(\zeta)(n^2 + 1)$ for $n \in Z_2(\zeta)$, $|n| \leq 4|\zeta|/\pi$. According to Lemma 3.3, the inequality $\operatorname{Re}(\zeta - n\pi)^2 \geq \frac{\pi^2}{8}(n^2 + 1)$ holds for every $|n| > 4|\zeta|/\pi$. Therefore the inequality

$$\operatorname{Re}(\zeta - n\pi)^2 \geq c(\zeta)(n^2 + 1) \quad (3.22)$$

holds for every $n \in Z_2(\zeta)$ with some $c(\zeta) > 0$ which does not depend on n . Hence

$$|H_\varepsilon^2(\zeta)| \leq \sqrt{\frac{\pi}{\varepsilon}} \sum_{n \in Z_2(\zeta)} e^{-c(\zeta)(n^2+1)/\varepsilon} \leq \sqrt{\frac{\pi}{\varepsilon}} \sum_{-\infty < n < \infty} e^{-c(\zeta)(n^2+1)/\varepsilon}. \quad (3.23)$$

Let us estimate the value in the right hand side of (3.23). Clearly

$$\sqrt{\frac{\pi}{\varepsilon}} \sum_{-\infty < n < \infty} e^{-c(\zeta)(n^2+1)/\varepsilon} = \sqrt{\frac{\pi}{\varepsilon}} e^{-c(\zeta)/\varepsilon} + 2\sqrt{\frac{\pi}{\varepsilon}} e^{-c(\zeta)/\varepsilon} \sum_{1 \leq n < \infty} e^{-c(\zeta)n^2/\varepsilon}$$

and

$$\sum_{1 \leq n < \infty} e^{-c(\zeta)n^2/\varepsilon} \leq \int_0^\infty e^{-c(\zeta)x^2/\varepsilon} dx = \frac{1}{2} \sqrt{\frac{\pi\varepsilon}{c(\zeta)}}.$$

Thus

$$|H_\varepsilon^2(\zeta)| \leq \sqrt{\frac{\pi}{\varepsilon}} e^{-c(\zeta)/\varepsilon} + \frac{\pi}{\sqrt{c(\zeta)}} e^{-c(\zeta)/\varepsilon}, \quad (3.24)$$

and (3.21) holds. \square

Lemma 3.5. *If $Z_1(\zeta) \neq \emptyset$, then*

$$\overline{\lim}_{\varepsilon \rightarrow +0} |H_\varepsilon^1(\zeta)| = \infty. \quad (3.25)$$

Proof. Given $\zeta \in \mathbb{C}$, the numbers $-(\zeta - n)^2$, where n runs over \mathbb{Z} , need not be pairwise different. However, for fixed $\zeta \in \mathbb{C}$, each number can appear among the numbers $\{-(\zeta - n)^2\}_{n \in \mathbb{Z}}$ not more than twice: the mapping $w \rightarrow (w - \zeta)^2$ covers the punctured plane $\mathbb{C} \setminus \zeta$ twice. Let p be the total number of pairwise different numbers $-(\zeta - n)^2$, $n \in Z_1(\zeta)$. Since $Z_1(\zeta)$ always is finite, $p < \infty$. Since $Z_1(\zeta) \neq \emptyset$, $p > 0$. Let λ_k , $1 \leq k \leq p$, be pairwise different representatives of the numbers $-(\zeta - n)^2$, $n \in Z_1(\zeta)$, the number λ_k appears in the set $-(\zeta - n)^2$, $n \in Z_1(\zeta)$, with multiplicity⁵ n_k , where n_k is either 1, or 2. Denoting $\tau = 1/\varepsilon$, we present the value $H_\varepsilon^1(\zeta)$ as

$$H_\varepsilon^1(\zeta) = \sqrt{\pi\tau} \sum_{1 \leq k \leq p} n_k e^{\lambda_k \tau}, \quad (3.26)$$

where the numbers λ_k , $1 \leq k \leq p$, are pairwise different, $\operatorname{Re} \lambda_k \geq 0$, and n_k is either 1, or 2. Since the factor $\sqrt{\tau}$ tends to ∞ as $\tau \rightarrow \infty$, it is enough to prove that

$$0 < \overline{\lim}_{\tau \rightarrow \infty} |T(\tau)| \leq +\infty, \quad (3.27)$$

where

$$T(\tau) = \sum_{1 \leq k \leq p} n_k e^{\lambda_k \tau}. \quad (3.28)$$

Lemma 3.5 is a consequence of the following fact.

Lemma 3.6. *Let $T(\tau)$ is a trigonometric polynomial of the form (3.28), where*

$$\operatorname{Re} \lambda_k \geq 0, \lambda_k \text{ are pairwise different, } n_k \neq 0, 1 \leq k \leq p. \quad (3.29)$$

Then the limiting relation (3.27) holds.

⁵So $\sum_{1 \leq k \leq p} n_k = |Z_1(\zeta)|$.

Proof. We order the numbers λ_k so that

$$\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = \cdots = \operatorname{Re} \lambda_q > \operatorname{Re} \lambda_{q+1} \geq \cdots \geq \operatorname{Re} \lambda_p \ (\geq 0).$$

Let

$$\lambda_k = \mu_k + i\nu_k, \quad \mu_k \in \mathbb{R}, \nu_k \in \mathbb{R}, \quad 1 \leq k \leq p.$$

Denote $\mu = \mu_1 = \mu_2 = \cdots = \mu_q$. Then

$$T(\tau) = e^{\mu\tau} (S(\tau) + R(\tau)), \quad (3.30)$$

where

$$S(\tau) = \sum_{1 \leq k \leq q} n_k e^{i\nu_k \tau}, \quad R(\tau) = \sum_{q+1 \leq k \leq p} n_k e^{(\mu_k - \mu)\tau} e^{i\nu_k \tau},$$

and

$$\mu \geq 0, \quad \mu > \mu_k \text{ for } q+1 \leq k \leq p, \quad \nu_k \in \mathbb{R} \text{ for } 1 \leq k \leq p, \quad (3.31)$$

Since $\mu_k < \mu$ for $q+1 \leq k \leq p$,

$$\lim_{\tau \rightarrow +\infty} R(\tau) = 0. \quad (3.32)$$

Since $n_k \neq 0$ and the numbers $\nu_k \in \mathbb{R}$ are pairwise different, the function $S(\tau)$ is an almost periodic (in particular, periodic) function, $S(\tau) \not\equiv 0$. Therefore

$$\overline{\lim}_{\tau \rightarrow \pm\infty} |S(\tau)| = \sup_{\tau \in \mathbb{R}} |S(\tau)| > 0. \quad (3.33)$$

The limiting relation (3.27) is a consequence of (3.30), (3.32), (3.33) and $\mu \geq 0$. \square

From definition of the divergence set \mathcal{V}_H (see Definition 3.13), from Lemmas 3.5, 3.4, and from the equality (3.18) we obtain the following description of the divergence set \mathcal{V}_H .

Theorem 3.

$$\mathcal{V}_H = \{\zeta \in \mathbb{C} : Z_1(\zeta) \neq \emptyset\}, \quad (3.34)$$

where $Z_1(\zeta)$ is defined in (3.20).

Now we obtain a geometric description of the divergency set \mathcal{V}_H . From this description Theorem 2 follows.

For $n \in \mathbb{Z}$, let

$$Q_n = \{\zeta \in \mathbb{C} : \operatorname{Re}(\zeta - n\pi)^2 > 0\}.$$

According to (3.20) and Theorem 3,

$$\mathbb{C} \setminus \mathcal{V}_H = \bigcap_{n \in \mathbb{Z}} Q_n. \quad (3.35)$$

It is clear that

$$Q_n = Q_0 + n, \quad \forall n \in \mathbb{Z},$$

where

$$Q_0 = \{\zeta \in \mathbb{C} : -\pi/4 < \arg \zeta < \pi/4\} \bigcup \{\zeta \in \mathbb{C} : 3\pi/4 < \arg \zeta < 5\pi/4\}.$$

Simple geometric considerations show that

$$\bigcap_{n \in \mathbb{Z}} Q_n = \bigcup_{n \in \mathbb{Z}} T_n, \quad (3.36)$$

where

$$T_n = T_0 + n, \quad n \in \mathbb{Z}, \quad (3.37)$$

and T_0 is the open square with the vertices at the points $\zeta = 0$, $\zeta = \pi$, $\zeta = i\pi/2$, $\zeta = -i\pi/2$. According to (3.35) and (3.36), the divergence set \mathcal{V}_H is

$$\mathcal{V}_H = \Gamma_+ \bigcup \Gamma_-, \quad (3.38)$$

where

$$\Gamma_+ = \{\xi + i\eta : -\infty < \xi < \infty, \quad \eta \geq \min_{n \in \mathbb{Z}} |\xi - n\pi|\}, \quad (3.39)$$

$$\Gamma_- = \{\xi + i\eta : -\infty < \xi < \infty, \quad \eta \leq -\min_{n \in \mathbb{Z}} |\xi - n\pi|\}. \quad (3.40)$$

According to (3.15),

$$\mathcal{V}_h = \exp\{2i\Gamma_+\} \bigcup \exp\{2i\Gamma_-\}. \quad (3.41)$$

It is clear that

$$\exp\{2i\Gamma_+\} \subset (\{1\} \cup \mathbb{D}^+), \quad \exp\{2i\Gamma_-\} \subset (\{1\} \cup \mathbb{D}^-). \quad (3.42)$$

From (3.9), (3.41) and (3.42) we conclude that the divergence set \mathcal{V}_f is

$$\mathcal{V}_f = \exp\{2i\Gamma_-\}. \quad (3.43)$$

It is clear that

$$\exp\{2i\Gamma_-\} = \mathbb{C} \setminus \mathcal{G},$$

where the set \mathcal{G} was defined in Definition 1. Thus

$$\mathcal{V}_f = \mathbb{C} \setminus \mathcal{G}. \quad (3.44)$$

Theorem 2 is proved. \square

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